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On Optimization of Approximate Integration over a Ball

V. F. Babenko, *Dnepropetrovsk*, *Ukraine*

S. V. Borodachov (speaker), *Vanderbilt*, *USA*

AMS Classification: 41A55, 41A63.

Keywords and phrases: cubature formula, class of functions, error over a class.

Let $d, n \in \mathbf{N}$, $d > 1$, $r > 0$, $B_d[\overline{a}, r] = \{x \in \mathbf{R}^d : |x - \overline{a}| \leq r\}$ and $S_d[\overline{a}, r] = \{x \in \mathbf{R}^d : |x - \overline{a}| = r\}$. Denote $b_d = \text{mes} B_d[\overline{0}, 1]$, where $\overline{0} = (0, \ldots, 0) \in \mathrm{R}^d$, and let σ_d be the $(d-1)$-dimensional measure of $S_d[\overline{0}, 1]$. Let $p : (0, r] \to (0, \infty)$ be a continuous function such that $\int_0^r p(t) t^{d-1} dt < \infty$. Denote by $V_d(p)$ the closure in the uniform metric on $B_d[\overline{0}, r]$ of the class of continuously differentiable functions $f : B_d[\overline{0}, r] \to \mathbf{R}$ such that $\int_{B_d[\overline{0}, r]} |\text{grad} f(x)|\, p(|x|) dx \leq 1$.

For a function $f \in V_d(p)$ put $F_\rho(f) = \sigma_d^{-1} \rho^{1-d} \int_{S_d[\overline{0}, \rho]} f(x) dS, \rho \in (0, r]$, $F_0(f) = f(\overline{0})$. Let U_n^d be the set of all cubature formulas $K : V_d(p) \to \mathbf{R}$ for approximate integration along $B_d[\overline{0}, r]$ of the form $K(f; \overline{r}_n, \overline{c}_n) = \sum_{k=1}^n c_k \cdot F_{r_k}(f)$, where $\overline{c}_n = (c_1, \ldots, c_n) \in \mathbf{R}^n$, $\overline{r}_n = (r_1, \ldots, r_n) \in \mathbf{R}^n$, and $0 \leq r_1 < \ldots < r_n \leq r$. Set

$$R_n\left(V_d(p)\right) = \inf_{\overline{r}_n, \overline{c}_n} \sup_{f \in V_d(p)} \left| \int_{B_d[\overline{0}, r]} f(x) dx - K(f; \overline{r}_n, \overline{c}_n) \right|. \tag{2}$$

Problem. It is required to find value (2) and optimal formulae for the class $V_d(p)$, i.e. sets $\overline{r}_n^*, \overline{c}_n^*$, delivering the infinum on the right-hand side of (2), if they exist.

Denote by $Z_{n,d}$ the set of all functions z such that $z(t) = 0$, $t \in [0, r_1)$, $z(t) = C_k$, $t \in [r_k, r_{k+1})$, $k = \overline{1, n-1}$, $z(t) = b_d r^d$, $t \in [r_n, r]$, where $C_k \in \mathbf{R}$, $k = \overline{1, n-1}$, $0 \leq r_1 < \ldots < r_n \leq r$. Let also $\|g\|_{p,\infty} = \text{vrai}\sup_{t \in [0, r]} \left| g(t) (p(t))^{-1} t^{1-d} \right|$.

Theorem *It is true that* $R_n(V_d(p)) = \sigma_d^{-1} \inf_{z \in Z_{n,d}} \|z(t) - b_d t^d\|_{p,\infty}$. *If* $p(t) = t^{1-d}$, *then formula* $K(\cdot; \overline{r}_n^*, \overline{c}_n^*) \in U_n^d$, *where* $c_k^* = b_d r^d n^{-1}$, $r_k^* = r\,(2k-1)^{1/d}(2n)^{-1/d}$, $k = \overline{1,n}$, *is optimal in the class* $V_d(p)$. *In addition,* $R_n(V_d(p)) = r^d(2nd)^{-1}$.

If $p(t) = 1$ *and* $d = 2$, *then* $c_k^* = 2\pi r^2 k(n(n+1))^{-1}$, $r_k^* = rk(n(n+1))^{-1/2}$, $k = \overline{1,n}$, *and* $R_n(V_2(p)) = 2^{-1} r(n(n+1))^{-1/2}$. *If* $d \geq 3$, $r_1^* = r\varphi(x_n)(1+\varphi(x_n))^{-1/d}$, $r_k^* = x_{k-1} r_{k-1}^*$, $k = \overline{2,n}$, $c_k^* = 2b_d\varphi(x_k)(r_k^*)^d$, $k = \overline{1,n}$, *and* $R_n(V_d(p)) = r_1^*/d$, *where* $x_k > 1$, $k = \overline{1,n}$, $\varphi(t) = (t^d - 1)(t^{d-1}+1)^{-1}$ *and* $\varphi(x_1) = 1$, $\varphi(x_k) = \varphi(x_{k-1})x_{k-1}^{-1}$, $k = \overline{2,n}$.